independent systems of differential equations by transformations of the independent variables. These two systems are solved simultaneously for arbitrary temperature distribution, transverse load, and uniform edge compression for the case of simply supported edges. Although the analysis of Grigolyuk is for a doubly curved shell, the curvature effect does not increase the difficulty of analysis appreciably, and, consequently, it would appear that the analysis due to Ebcioglu possesses certain general features lacking in that due to Grigolyuk.

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Investigation of Nonlinear Systems Containing Stages with Variable Time Constant

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The analysis and synthesis of nonlinear systems with ordinary types of nonlinearity, such as Coulomb-friction, backlash or the bang-bang type of nonlinearities, can be carried out fairly simply by Popov's method. For some typical nonlinearities, harmonic-linearization coefficients have been determined, the use of which simplifies calculations considerably. In a number of cases, however, other forms of nonlinearities are encountered, for which harmonic-linearization formulas have not been obtained. Therefore, the investigation of automatic systems with such nonlinearities can entail difficulties. The present paper develops a method for the study of systems with such a nonlinearity, for which the time constant is not a constant parameter.

STAGES with time constants that vary with the operating conditions are often met in engineering analyses and syntheses of systems. Examples include numerous circuits for extracting the d.c. component of a signal, where the charging and discharging of capacitors occur through different circuits. In Fig. 1, for example, the capacitor C is charged from the cathode follower T-1 via the resistor r_1, but is discharged via the resistor r_2. Here, the charging and discharging time constants are, in general, substantially different. In engineering practice, the discharging time constant is made, as a rule, considerably larger. A similar situation, though even more complicated, is met in experimental investigations of the transients in magnetic amplifiers with positive feedback.

A typical feature in all such stages is the fact that the time constant of the stage depends, in the first place, on the sign of the variation of the output quantity. As a rule, the time constant is small for increasing signal and is large for decreasing signal.

The transient behavior in stages with variable time constant is illustrated in Fig. 2. Here, the stage behaves as if it possessed two time constants T_1 and T_2. The subsequent analysis requires that we write fairly accurately the equation of such a stage, and that the form of the equation should reproduce as closely as possible the essential physical features of the nonlinearity.

The equation of a stage with a transient similar to that shown in Fig. 2 can be written in various ways. However, its most convenient form is

$$[1 + (1 + c_1 \operatorname{sign} p|U_2|)Tp]U_2 = kU_1 \tag{1}$$
$$T = (T_1 + T_2/2)$$

where U_1 and U_2 are input and output quantities, respectively; T the mean value of time constant; and c_1 a coefficient determined by amplitude of variation of time constant.

More precisely, c_1 is defined by the expression $T(1 + c_1) = T_1$, $T(1 - c_1) = T_2$:

$$c_1 = \frac{(T_1/T_2) - 1}{(T_1/T_2) + 1} \tag{2}$$

if $T_1 < T_2$, c_1 is a negative quantity. It is of interest to point out that the time constant T, i.e., the mean value of the real time constants T_1 and T_2, does not exist physically. Equation (1) expresses physically the fact that as U_2 increases, the value of sign $p|U_2|$ is equal to $+1$, that is, the value of c_1 is added algebraically to unity, and the growth process is governed by the time constant $T_1 = T(1 + c_1)$.

When the output quantity U_2 decreases, then sign $p|U_2| = -1$, c_1 is subtracted from unity, and the time constant of the transient equals $T_2 = T(1 - c_1)$.

Equation (1) can be rewritten

$$(1 + Tp)U_2 + (\operatorname{sign} p|U_2|)c_1TpU_2 = kU_1 \tag{3}$$

It is evident that Eq. (3) is essentially nonlinear, since one of its terms is the product of the rate of variation of the output quantity, pU_2, and the sign of the variation of the modulus of the output quantity, sign $p|U_2|$.

Harmonic linearization of the nonlinearity requires that we expand the nonlinear function into a Fourier series, i.e., that we represent it as

$$y = (\operatorname{sign} p|U_2|)c_1TpU_2 = \sum_{k=1}^{\infty} A_k \sin k\omega t + \sum_{k=1}^{\infty} B_k \cos k\omega t \tag{4}$$

Here, the coefficients A_1 and B_1, which determine the amplitude of the first harmonic of the oscillations, can be evaluated from the formulas

$$A_1 = \frac{1}{\pi}\int_0^{2\pi} y(t) \sin\omega t d\omega t$$
$$B_1 = \frac{1}{\pi}\int_0^{2\pi} y(t) \cos\omega t d\omega t \tag{5}$$

Translated from Izvestiia Akademii Nauk SSSR, OTN, Energetika i Avtomatika (Bulletin of the Academy of Sciences of the USSR, Division of Technical Sciences, Power and Automation) 5, 89–95 (1962). Translated by Scripta Technica, Inc., New York.

The determination of the value of A_1 from the first of Eqs. (5) reduces to evaluating the following definite integral:

$$A_1 = \frac{4}{\pi}\int_0^{\pi/2} Tc_1 pU_2 \sin\omega t\omega t \qquad (6)$$

Under self-oscillatory conditions we can write, for U_2:

$$U_2 = a \sin\omega t$$

where a is the amplitude of the first harmonic of the self-oscillations and ω is the self-oscillation frequency. We have, now, $pU_2 = a\omega \cos\omega t$ and the first of Eqs. (5) assumes the form

$$A_1 = \frac{4}{\pi}\int_0^{\pi/2} Tc_1 a\omega \cos\omega t \sin\omega t d\omega t \qquad (7)$$

or, after some manipulation

$$A_1 = \frac{Tc_1 a\omega}{\pi}\int_0^{\pi/2} \sin 2u d2u = \frac{2Tc_1 a\omega}{\pi} \quad (u = \omega t)$$

As can be easily verified, $B_1 = 0$. Thus, we can write for the nonlinearity y the expression

$$y = (2Tc_1\omega/\pi)U_2 \qquad (8)$$

By substituting the expression obtained for the harmonically linearized nonlinearity into Eq. (1), we finally obtain the equation of the stage

$$[1 + Tp + (2Tc_1\omega/\pi)]U_2 = kU_1 \qquad (9)$$

In the subsequent analysis we shall use Eq. (9) rewritten in the form

$$\frac{U_2}{U_1} = \frac{k}{1 + (2Tc_1\omega/\pi)} \frac{1}{1 + \{1/[1 + (2Tc_1\omega/\pi)]\}Tp} \qquad (10)$$

The following conclusions can be drawn from an analysis of Eq. (10):

The nonlinearity of a stage that exhibits a different time constant for increasing and decreasing signal reduces, by harmonic linearization, to an equation in which not only the time constant but also the gain of the stage are modified.

The gain and the time constant of the harmonically linearized stage depend only on the frequency of the oscillations and not on their amplitude.

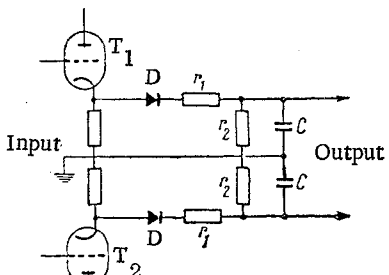

Fig. 1.

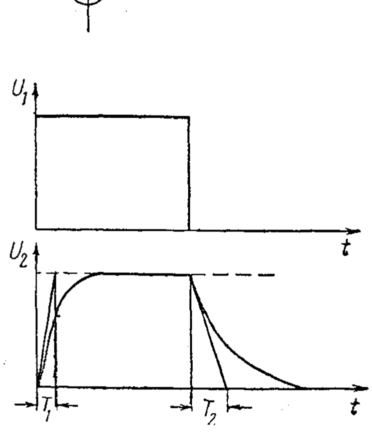

Fig. 2.

When the time constant for increasing signal is smaller than that for decreasing signal ($c_1 < 0$), the equivalent gain $k' = k/(1 + 2Tc_1\omega/\pi)$ and the equivalent time constant $T' = T/(1 + 2Tc_1\omega/\pi)$ are increased.

Since the expression for the transfer function of the harmonically linearized stage considered does not involve, in contrast to ordinary nonlinearities, the oscillation amplitude, then limit cycle oscillations, in the usual sense of this term, may be absent.

A system that comprises such a stage, having reached, under an external action, the stability boundary, will begin to oscillate with an amplitude corresponding to the value of the external action. Such a system will essentially behave as a linear system, except that the gain and the time constant of one of its stages will depend (and will depend substantially) on the parameters of the other stages comprised in the system.

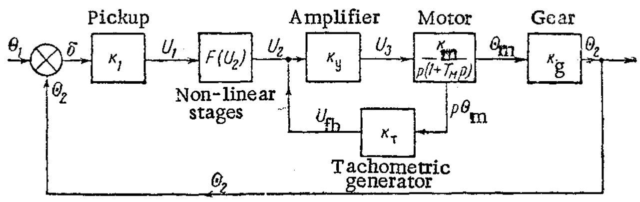

Fig. 3.

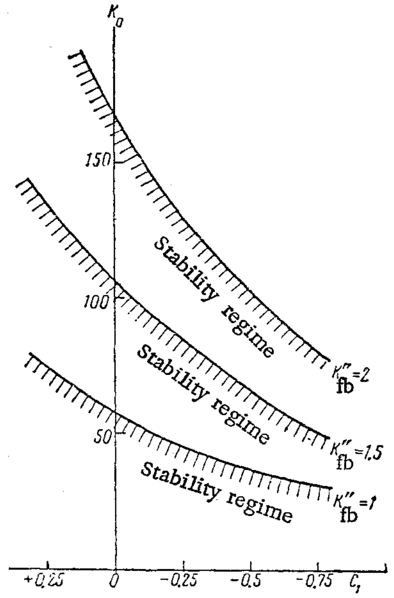

Fig. 4.

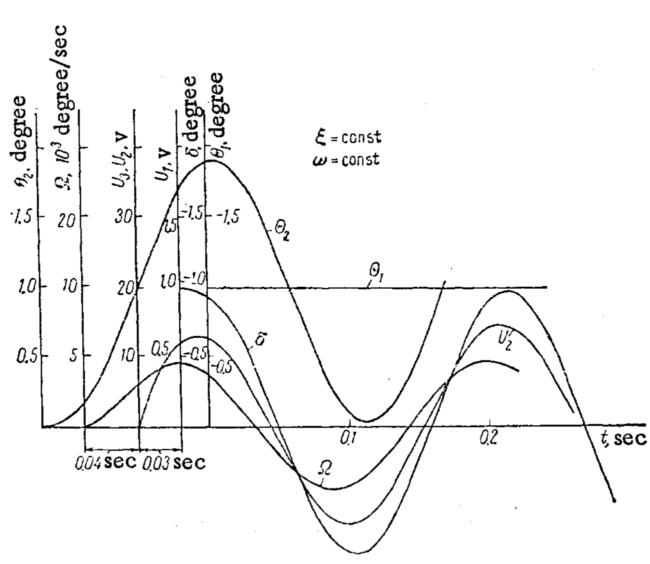

Fig. 5.

This is explained by the fact that the frequency of the oscillations of the system is determined by the parameters of all stages. A method of investigating systems with such nonlinearity will be considered on the basis of a concrete example. The block diagram of a typical system is shown in Fig. 3. The dynamics of the transients of the system of Fig. 3 is described by the system of equations

$$\delta = \theta_1 - \theta_2 \qquad U_{fb} = k_T p \theta_m$$

$$(1 + T_M p) p \theta_m = k_m U_3 \qquad U_1 = k_1 \delta \tag{11}$$

$$\theta_2 = k_g \theta_m \qquad \left(1 + \frac{\pi T}{\pi + 2Tc_1\omega} p\right) U_2 = \frac{k\pi}{\pi + 2Tc_1\omega} U_1$$

$$U_3 = k_a(U_2 - U_{fb})$$

After some manipulations we obtain the equation for the error of the harmonically linearized system

$$\delta = \frac{(T_M p + 1 + k_a k_m k_T)\left(1 + \frac{T\pi}{\pi + 2Tc_1\omega} p\right)}{[(T_M p + 1)p + k_a k_m k_T p]\left(1 + \frac{T\pi}{\pi + 2Tc_1\omega} p\right) + \frac{k_0 \pi}{\pi + 2Tc_1\omega}} p\theta_1 \tag{12}$$

The characteristic equation of the system has the form

$$A_0 p^3 + A_1 p^2 + A_2 p + A_3 = 0 \tag{13}$$

where

$$A_0 = 1$$

$$A_1 = \frac{1}{T}\left(1 + \frac{2Tc_1\omega}{\pi}\right) + \frac{k_{fb}''}{T_M}$$

$$A_2 = \frac{k_{fb}''}{T_M T}\left(1 + \frac{2Tc_1\omega}{\pi}\right) \tag{14}$$

$$A_3 = \frac{k_0}{T_M T}$$

$$k_0 = k_1 k_a k_m k_g k$$

$$k_{fb}'' = 1 + k_a k_m k_T$$

The system is found on the stability boundary if $A_1A_2 - A_0A_3 = 0$. Therefore

$$k_0 = k_{fb}''\left[\frac{1}{T}\left(1 + \frac{2Tc_1\omega}{T}\right) + \frac{k_{fb}''}{T_M}\right]\left(1 + \frac{2Tc_1\omega}{\pi}\right) \tag{15}$$

The first problem to be solved to study the system is the determination of the effect of the nonlinearity amplitude (the effect of the range of variation of the time constant) on the value of the critical gain of the system. We cannot, however, determine k_0 as a function of c_1 from Eq. (15) since ω, the frequency of the oscillations of the system, also occurs in this equation.

To solve the problem, we can use, in addition to (15), the equation:[1]

$$\omega^2 = \frac{A_3}{A_1} = \frac{\pi k_0}{T_M(\pi + 2T_{c1}\omega) + Tk_{fb}''\pi} \tag{16}$$

Thus, to plot k_0 as a function of c_1, we have the system of equations (15) and (16). The most convenient procedure consists in plotting k_0 as a function of ω both according to (15) and (16) on one and the same graph for various values of c_1. The intersection of curves corresponding to one and the same value of c_1 give $k_0 = k_0(c_1)$. In Fig. 4, curves of $k_0(c_1)$ have been plotted for the system of Fig. 3 for particular values of the parameters: $T_M = 0.04$ and $T = 0.03$ sec, and for various values of the parameter $k_{fb}'' = 1$, 1.5, and 2. An analysis of the curves of Fig. 4 suggests the following conclusions:

The presence of nonlinearity with $c_1 < 0$ reduces the critical gain of the system according to an approximate linear law.

The introduction of a nonlinearity in the system, i.e., the replacement of a stage with constant time constant by a stage with a variable time constant, larger for increasing signal than for decreasing signal ($c_1 > 0$), can nearly double the critical gain of the system. This result is explained physically by the fact that, when oscillations arise in the system, the gain and the time constant of the nonlinear stage with such nonlinearity automatically decrease [see Eq. (10)] which naturally promotes conditions for the stopping of oscillations.

To verify the correctness of the conclusions drawn we have plotted, using Bashkirov's method, the transients in a linear (Fig. 5) and a nonlinear (Fig. 6) system of the same gain ($k_0 = 60$ sec^{-1}). The system is found in the first case on the stability boundary, whereas in the second case, in agreement with the calculations carried out (Fig. 4), oscillations in it are attenuated.

However, of the greatest interest for engineering design is the investigation in nonlinear systems of the transients. The most useful method for this is Popov's method.[1] We shall seek the transient in the form

$$x = a_0 e^{\int_0^t \xi(t)dt} \sin \int_0^t \omega(t) dt \tag{17}$$

where ξ and ω are, respectively, the attenuation coefficient and the frequency of the oscillations, i.e., the basic parameters of the transient, which in the general case of nonlinear systems are functions of the amplitudes of the oscillations Ω. In many particular cases, when the coefficients of the harmonically linearized characteristic equation of the system depend only on the amplitude of the oscillations, the problem of investigating the transients in the system is solved com-

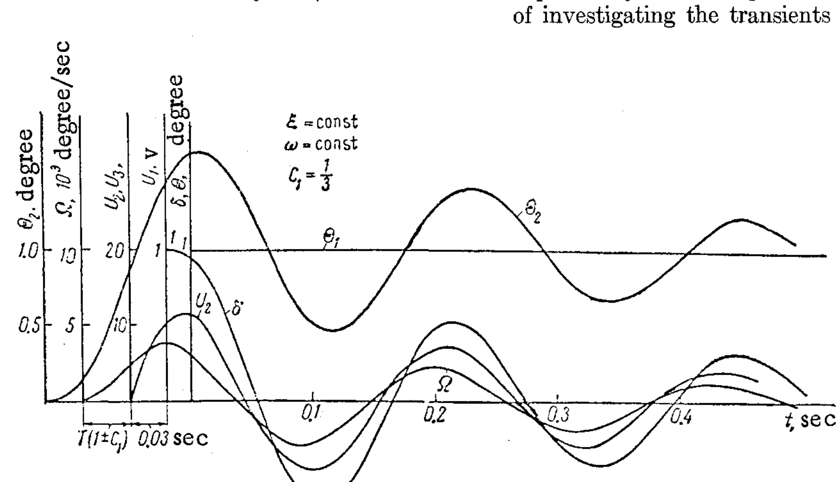

Fig. 6.

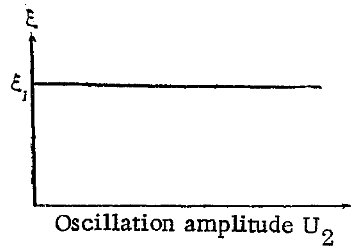

Fig. 7.

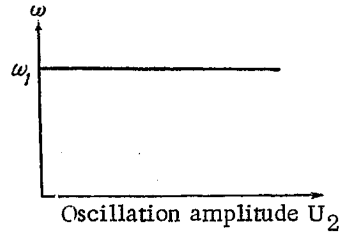

Fig. 8.

paratively easily.[1] In our case some coefficients of Eq. (13) are functions of the frequency of the oscillations, and the method of investigation is somewhat more complicated.[2]

The equation for determining the attenuation coefficient ξ can be written in the form:[2]

$$\xi = -\frac{A_1A_2 - A_3}{2[A_2 + (A_1 + 2\xi)^2]} = -\frac{\left[\frac{1}{T}\left(1 + \frac{2Tc_1\omega}{\pi}\right) + \frac{k_{fb}''}{T_M}\right]\left(1 + \frac{2Tc_1\omega}{\pi}\right)k_{fb}'' - k_0}{2\left[k_{fb}''\left(1 + \frac{2Tc_1\omega}{\pi}\right) + TT_M\left[\frac{1}{T}\left(1 + \frac{2Tc_1\omega}{\pi}\right) + \frac{k_{fb}''}{T_M} + 2\xi\right]^2\right]} \tag{18}$$

The attenuation coefficient ξ depends here on the system parameters and on the oscillation frequency ω, and we need, therefore, an additional equation for the oscillation frequency:[2]

$$\omega^2 = \frac{A_3}{A_1 + 2\xi} - \xi^2 = \frac{k_0}{T_M\left(1 + \frac{2Tc_1\omega}{\pi}\right) + k_{fb}''T + 2T_MT\xi} - \xi^2 \tag{19}$$

The solution of Eqs. (18) and (19), i.e., the determination of ξ and ω for given system parameters, is conveniently carried out by a method of successive approximations. As a first approximation, we can in this case neglect the quantities ξ and $2Tc_1\omega/\pi$ in (19) in comparison with other terms. Equation (19) provides, after this simplification, a first approximation for the oscillation frequency ω. By substituting the value obtained for ω in the right-hand side of (19) we can determine a second preliminary approximation of the oscillation frequency ω. This value of ω can be substituted in the right-hand side of Eq. (18) which, in turn, must be solved by a method of successive approximations. This is done by first putting $\xi = 0$ in the right-hand side of (18), solving for ξ, substituting this approximation for ξ in the right-hand side of (18), and solving for ξ again. The approximations obtained for ξ and ω are now substituted into (19) to give the next approximation for ω.

The process must be repeated until a good agreement is found between the last and the penultimate approximations. In practice, two or three approximation steps are sufficient.

An analysis of the exact curves of the transients (Figs. 5 and 6) and of graphs (Figs. 7 and 8) plotted from results of approximate calculations by the method of harmonic linearization enables us to conclude that, for the type of nonlinearities considered, we can no longer use the usual picture of transients in nonlinear systems, in the form of an exponential with a variable attenuation coefficient and a variable oscillation frequency.

The calculation carried out has shown that cases are possible when the transient in a nonlinear system is governed by a constant attenuation coefficient ξ and constant oscillation frequency ω.

—Submitted March 2, 1962

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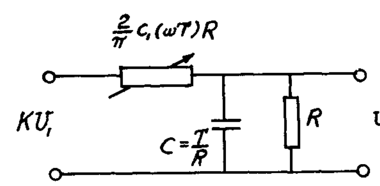
[2] Khlypalo, Ye. P., "Approximate investigations of transients in non-linear system of second class," Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk (Bull. Acad. Sci. USSR, Div. Tech. Sci.), No. 10 (1958).

Reviewer's Comments

This paper applies the describing function method of nonlinear system analysis to a form of nonlinearity that occurs frequently in feedback control systems, usually for incidental rather than purposeful reasons. It is more than merely another mathematical exercise in the use of the describing function because it leads to conclusions with practical implications. Simply stated, these conclusions are:

1) The use of a coupling stage similar to the author's Fig. 1 can have a detrimental effect on system stability and degree of stability. This will be true of all such capacitive coupling elements when characterized by longer time constants under discharging conditions than under charging conditions.

2) A modification of such capacitive coupling stages so that the time constant under charging conditions will be the longer would enhance the stability of the system.



This is not to say that one should introduce such a capacitive coupling stage purposely to enhance stability. Often, however, such a storage capacitor as in the author's Fig. 1, or its analog in other physical forms, is required for other purposes (such as demodulation of carrier signals or the holding of sampled data). Under such circumstances, the author recommends in his conclusions a design having a longer time constant under charging conditions.

From a theoretical standpoint, the problem considered by the author is interesting because it lends to an unusual type of describing function which is amplitude independent but frequency dependent. The effect is similar to that of a simple RC coupling circuit as shown here.

When condition (2) applies, system stability is enhanced in a manner somewhat similar (qualitatively) to that effected by "tachymetric limiting."[1]

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[1] Kochenburger, R. J., "Limiting in feedback control systems," Trans. Am. Inst. Elec. Engrs. **72**, 180–194 (1953).